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Internal waves of permanent form in fluids of great depth

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This paper presents a general theoretical treatment of a new class of long stationary waves with finite amplitude. As the property in common amongst physical systems capable of manifesting these waves, the density of the (incompressible) fluid varies only within a layer whose thickness h is much smaller than the total depth, and it is h rather than the total depth that must be considered as the fundamental scale against which wave amplitude and length are to be measured. Internal-wave motions supported by the oceanic thermocline appear to be the most promising field of practical application for the theory, although applications to atmospheric studies are also a possibility.

The waves in question differ in important respects from long waves of more familiar kinds, and in § 1 their character is discussed on the basis of a comparison with solitary-wave and cnoidal-wave theories on customary lines, such as apply to internal waves in fluids of limited depth. A summary of some simple experiments is included at the end of § 1. Then, in § 2, the comparatively easy example of two-fluid systems is examined, again to illustrate principles and to prepare the way for the main analysis in § 3. This proceeds to a second stage of approximation in powers of wave amplitude, and its leading result is an equation (3·51) determining, for arbitrary specifications of the density distribution, the form of the streamlines in the layer of heterogeneous fluid. Periodic solutions of this equation are obtained, and their properties are discussed with regard to the interpretation of internal bores and wave-resistance phenomena. Solutions representing solitary waves are then obtained in the form

$$f(x) = a\lambda^2/(x^2+\lambda^2),$$

where x is the horizontal co-ordinate and where $a\lambda = O(h^2)$. (The latter relation between wave amplitude and length scale contrasts with the customary one, $a\lambda^2 = O(h^3)$). The main analysis is developed with particular reference to systems in which the heterogeneous layer lies on a rigid horizontal bottom below an infinite expanse of homogeneous fluid; but in § 4 ways are given to apply the results to various other systems, including ones in which the heterogeneous layer is uppermost and is bounded by a free surface. Finally, in § 5, three specific examples are treated: the density variation with depth is taken, respectively, to have a discontinuous, an exponential and a 'tanh' profile.

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1. Introduction

This investigation is concerned with waves of finite amplitude in stable heterogeneous-fluid systems of the general kind illustrated in figure 1. The distinguishing feature of these theoretical models is that the density variations extend only over a limited depth h but the total depth of the fluid is infinite. The first example, figure 1 (*a*), may be useful for the description of wave motions in the atmosphere, being an attractive alternative to models in which a rigid upper

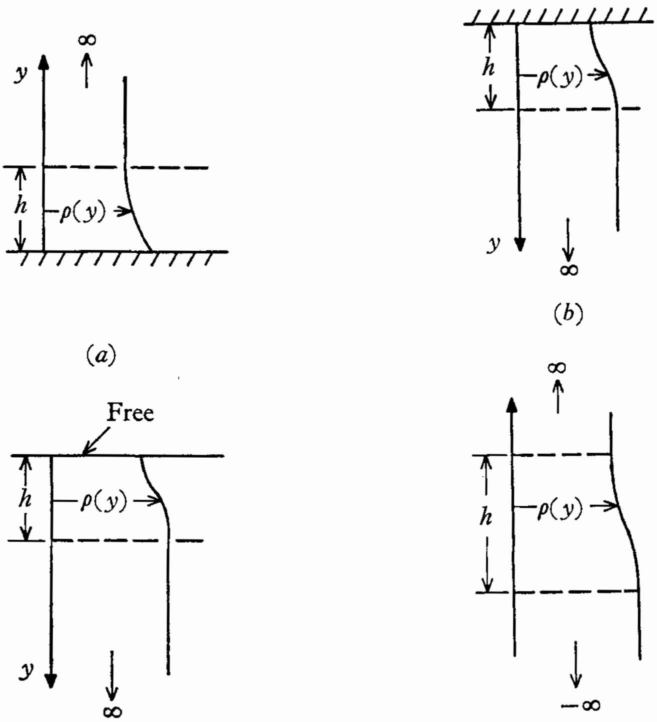

FIGURE 1. Illustration of various heterogeneous-fluid systems in general category under investigation.

boundary is supposed, and the examples in figures 1 (*c*) and 1 (*d*) are obviously relevant to the oceanic thermocline. In the analysis to be made, the lengths of the waves are assumed to be considerably greater than h, but of course no long-wave approximation is possible for the motion in the infinite expanse of homogeneous fluid that adjoins the heterogeneous layer. The latter aspect of the problem presents an analytical contingency unfamiliar in long-wave theories, such as the classical theory of long surface waves, and some novel results are forthcoming. In particular, a new class of solitary waves is found which promises to have an important bearing on the interpretation of internal waves in the oceans.

While the subject of the present study appears to have been unexplored previously in the literature,† a great deal of work has been done on internal solitary waves in fluids of finite total depth. Keulegan (1953) and Long (1956) pioneered this field by investigating solitary waves in two-layer systems, and the theory of solitary waves in fluids whose density varies continuously with height has been developed by Peters & Stoker (1960), Ter-Krikorov (1963), Long (1965), Benjamin (1966) and several others. The periodic cnoidal-wave solutions that are generally obtainable by the same methods of approximation that yield solitary waves (e.g. see Lamb 1932, § 253) were noted by most of these authors, and a complete classification of the physical applications of the theory was given by Benjamin (1966). However, the finite depth of the fluid, implying the influence of both lower and upper boundaries, is an essential specification of the theoretical models treated in all this work, and the results are useless for application to the present type of model. Consider, for instance, Long's results for a solitary wave in a two-layer system (or see Benjamin 1966, § 4, Example 2). If in these results the depth of one layer is made infinite, the wave speed appears in the limit to take a definite value in terms of the depth of the other layer, but the length of the wave appears to increase without bound. One might thus be misled into supposing there is no realizable solitary wave in a fluid of infinite total depth. But this is a spurious conclusion, of course, relating to a situation where the former methods of analysis break down.

The essentials of the present theory, and the contrast with customary solitary-wave theories as exemplified by the work just mentioned, may be explained simply as follows. The key to the interpretation is the 'dispersion relation' between the frequency ω and wave-number k of *infinitesimal* periodic waves, for which every dependent variable takes the form

$$u = \hat{u}\, e^{i(\omega t - kx)}, \tag{1.1}$$

where x is the co-ordinate in the direction of propagation and $\hat{u}$ may be a function of the co-ordinate perpendicular to x. In the customary case, the distinguishing property is that the phase velocity $c = \omega/k$ has a smooth maximum c_0 at $k = 0$, that is, for waves of extreme length. This means that, for small enough values of k, one has

$$\omega = kc_0(1-\beta k^2), \tag{1.2}$$

in which β is a positive constant, depending of course on the particular system under investigation. [Recall, for example, the well-known dispersion relation for surface waves on water of depth H,

$$\omega = (gk \tanh kH)^{\frac{1}{2}},$$

which for $kH \ll 1$ can be approximated in the form (1.2) with $c_0 = (gH)^{\frac{1}{2}}$ and $\beta = \frac{1}{6}H^2$.] By Fourier's theorem it follows from (1·2) that a long infinitesimal

† But while writing this paper the author became aware that work on the same problem has been done by Mr R. E. Davis and Prof. A. Acrivos at Stanford University. In particular, they have made some excellent experiments on internal solitary waves in fluids with density distributions as shown in figure 1 (*d*). It is hoped their work will be published soon, complementing the present independent study.

wave of general form propagating in the positive x-direction is governed approximately by the equation

$$\frac{\partial u}{\partial t}+c_0\frac{\partial u}{\partial x}+c_0\beta\frac{\partial^3 u}{\partial x^3}=0, \tag{1.3}$$

since this is satisfied independently by every component (1.1) with k sufficiently small.

Still referring to the customary case in general, we next note that if the effects of frequency dispersion are ignored, the non-linear effects of finite amplitude can usually be analysed without further approximation by well-tried procedures such as the 'shallow-water' theory for surface waves (e.g. see Stoker 1957, ch. 10). For a wave advancing towards a region at rest in which $u=0$, it appears that the dependent variable u is conserved along a characteristic given to a first approximation by $dx/dt = c_0+Cu$, where C is a constant, so that u satisfies

$$\frac{\partial u}{\partial t}+c_0\frac{\partial u}{\partial x}+Cu\frac{\partial u}{\partial x}=0. \tag{1.4}$$

Now, various schemes might be used to derive a consistent first approximation to the effects of *both* dispersion and finite amplitude. But the outcome of such a derivation may confidently be anticipated from what has already been noted: the last term in (1.3), representing the effects of dispersion, will simply be added to the last term in (1.4) which accounts for the non-linear effects. Thus one has

$$\frac{\partial u}{\partial t}+c_0\frac{\partial u}{\partial x}+Cu\frac{\partial u}{\partial x}+c_0\beta\frac{\partial^3 u}{\partial x^3}=0. \tag{1.5}$$

This type of equation was first introduced in the theory of long surface waves by Korteweg & de Vries (1895), and it is often named after them. Its relevance to internal waves in fluids of limited depth has been demonstrated explicitly by Benney (1966), having been strongly implied by much of the previous work on the subject.

The solutions of (1.5) in the form $u=f(x-ct)$ are very well known. There are periodic solutions expressible in terms of Jacobian elliptic functions—the cnoidal waves discovered by Korteweg & de Vries; and there is the solitary-wave solution

$$\left.\begin{aligned} u &= a\,\mathrm{sech}^2\{(aC/12c_0\beta)^{\frac{1}{2}}(x-ct)\},\\ \text{with}\qquad a &= 3(c-c_0)/C.\end{aligned}\right\} \tag{1.6}$$

Note the important properties (i) that the departure of the wave speed c from the maximum speed c_0 of infinitesimal waves is proportional to the amplitude a, and (ii) that the effective length scale of the wave, say λ, is $(12c_0\beta/aC)^{\frac{1}{2}}$. In terms of dimensionless variables, when for a particular system the units of length and time have been appropriately chosen (e.g. H and $H/c_0 = (H/g)^{\frac{1}{2}}$ for surface waves on water of depth H), it will turn out that $(12c_0\beta/C) = O(1)$; thus, the property (ii) means that $a\lambda^2 = O(1)$.

For systems of the type now to be treated, on the other hand, the dispersion relation for infinitesimal long waves has leading terms in the form

$$\omega = kc_0(1-\gamma|k|), \tag{1.7}$$

with $\gamma > 0$, and the difference between (1.7) and (1.2) has crucial implications. A Fourier component like (1.1) now satisfies

$$\frac{\partial u}{\partial t}+c_0\frac{\partial u}{\partial x} = c_0\gamma\,|k|\,\frac{\partial u}{\partial x}, \tag{1.8}$$

but obviously we cannot as before find a differential equation that will govern an infinitesimal wave of general form. However, assuming a wave that vanishes at $x = \pm\infty$, we may generalize (1.8) by the Fourier integral theorem, writing

$$\frac{\partial u}{\partial t}+c_0\frac{\partial u}{\partial x} = c_0\gamma\mathscr{F}\left\{\frac{\partial u}{\partial x}\right\}, \tag{1.9}$$

where $\mathscr{F}$ denotes the linear operation defined by

$$\mathscr{F}\{\xi(x)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}|k|\,e^{-ikx}\left\{\int_{-\infty}^{\infty}\xi(x')\,e^{ikx'}\,dx'\right\}dk. \tag{1.10}$$

[Note that $\mathscr{F}\{\xi(x)\}$ is the same as $\psi_y(x,0)$ if $\psi(x,y)$ is the solution to the Dirichlet problem for the upper half-plane: $\nabla^2\psi = 0$ in $y \geqslant 0$, $\psi = -\xi(x)$ on $y = 0$. Hence it can be seen that $-\mathscr{F}\{\xi(x)\}$ is the hydrodynamic pressure on an infinitesimal hump described by $y = \xi(x)$, over which there is an irrotational flow of an infinite fluid with unit density and velocity. This physical meaning is, of course, precisely the one underlying the appearance of $\mathscr{F}$ in the present analysis.] It may be anticipated that the effects of finite amplitude will enter in the same way as before, and so we are faced with the equation

$$\frac{\partial u}{\partial t}+c_0\frac{\partial u}{\partial x}+Cu\frac{\partial u}{\partial x} = c_0\gamma\mathscr{F}\left\{\frac{\partial u}{\partial x}\right\}, \tag{1.11}$$

which is the counterpart of the Korteweg–de Vries equation in the present case.

Assuming a solution depending on $\tilde{x} = x-ct$ alone, and using the fact that the operation of integration from $\tilde{x} = -\infty$ (where $u = 0$) and the operation $\mathscr{F}$ are commutative, one may obtain from (1.11)

$$-2(c-c_0)\,u+Cu^2 = 2c_0\gamma\mathscr{F}\{u\}. \tag{1.12}$$

This form of equation is derived on a more definite basis in the following pages, and it makes a nice conclusion to the analysis that the equation can be solved exactly. The solution is

$$\left.\begin{aligned} u &= \frac{a\lambda^2}{\tilde{x}^2+\lambda^2},\\ \text{with}\qquad a &= \frac{4(c-c_0)}{C} = \frac{4c_0\gamma}{C\lambda}.\end{aligned}\right\} \tag{1.13}$$

This new type of solitary wave has the property (i) noted above with regard to the customary case, but not the property (ii). Instead of the assumption $a\lambda^2 = O(1)$ which is the customary formal basis of solitary-wave derivations (cf. Ursell 1953), we must take $a\lambda = O(1)$. Periodic solutions, for which the definition (1.10) of $\mathscr{F}$ needs to be replaced by a corresponding expression involving Fourier series, will also be presented in § 3.

The writer has made some simple experiments on solitary waves of the new kind. Their intention was merely to check, for his own satisfaction, whether these waves are easily realizable, but the outcome was definite enough to seem worth putting on record. A Lucite tank 6 ft. long and 7 in. wide was filled with brine to a depth of about 6 in., and then a layer of salt-free but faintly dyed water about $\frac{1}{2}$ in. deep was carefully added. Straight-crested waves were generated by the horizontal movement of a wooden cylinder immersed in the upper layer and spanning the tank close to one end. The cylinder was supported by a frame which could slide freely along the edges of the tank, and an effective way to produce the waves was simply to push this through a few inches towards the opposite end of the tank. It was observed that from a fairly vigorous initial disturbance, a solitary wave would emerge distinctly and would travel to the far end of the tank, from which it was reflected intact. The interface between the brine and the fresh water was always displaced downwards by the wave, as is predicted by the theoretical results given below in §§ 4.1, 4.2 and in Example 1 of § 5; and the wave clearly showed the same remarkable property of persistence that is familiarly observed in practical examples of the classical solitary wave. As perhaps the most convincing evidence that this was indeed a non-linear wave of permanent form, it was found that attempts to produce waves of elevation—by moving the cylinder in the opposite direction—resulted only in transients which were very rapidly dispersed. An undesirable feature left unremedied in these simple experiments was that the free surface did not follow the internal-wave motions, being apparently immobilized by contamination; thus, small particles in the surface were seen to remain at rest, whereas particles floating just below it were seen to be displaced horizontally by the passage of a solitary wave. Consequently, through the action of viscosity in the boundary layer formed beneath the free surface, the waves were attenuated rather rapidly. For a reliable comparison with the inviscid-fluid theory, either experiments on a considerably larger scale would be needed, or special measures would have to be taken to clean the free surface.

2. Preliminary discussion of two-fluid systems

Before embarking on the main analysis in § 3, we examine the following easy example in order to illustrate principles. As depicted in figure 2, fluid of constant density ρ_1 lies, when undisturbed, in a uniform layer of depth h_1 below fluid of constant density ρ_2 ($<\rho_1$) in a layer of depth h_2, and the system is bounded at the bottom $y = 0$ and top $y = h_1 + h_2$ by rigid horizontal planes.

For infinitesimal waves propagating horizontally relative to a state of rest, such that the equation of the disturbed interface between the two fluids is

$$y = h_1 + \delta\, e^{ik(ct-x)}, \tag{2.1}$$

the relation between phase velocity c and wave-number k is easily found by the use of velocity potentials. The result is

$$c^2 = \frac{g(\rho_1 - \rho_2)}{k(\rho_1 \coth kh_1 + \rho_2 \coth kh_2)} \tag{2.2}$$

(cf. Lamb 1932, p. 371). According to (2.2), c^2 has a maximum at $k = 0$, its value being

$$c_0^2 = \frac{g(\rho_1 - \rho_2)}{(\rho_1/h_1) + (\rho_2/h_2)}, \tag{2.3}$$

and we note that c_0 remains determinate if we take $h_2 \to \infty$ in (2.3). Indeed, it appears that for waves of extreme length compared with h_1 an infinitely deep upper fluid has no inertial effect and exerts only a hydrostatic pressure on the perturbed interface, reducing the action of gravity in the ratio $(\rho_1 - \rho_2):\rho_1$. We go on to show, however, that the upper fluid always has a crucial influence on the *dispersive* properties of waves, at small but finite values of k.

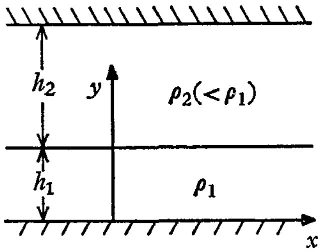

FIGURE 2. System comprising two immiscible fluids in superposed layers.

If h_1 and h_2 are both finite, it follows from (2.2) that

$$c = c_0\left\{1 - \tfrac{1}{6}k^2\left(\frac{\rho_1 h_1^2 h_2 + \rho_2 h_2^2 h_1}{\rho_1 h_2 + \rho_2 h_1}\right) + O(k^4 h_1^2 h_2^2)\right\}. \tag{2.4}$$

Thus, in this case we have an instance of the kind of dispersion relation for small k described by (1.2), and from this fact alone it can be concluded that solitary waves of the customary type are possible [for explicit demonstrations of them, see Keulegan (1953), Long (1956) or Benjamin (1966, §4, Example 2)]. If h_2/h_1 is now made very large, the coefficient of k^2 in (2.4) becomes proportional to $h_1 h_2$; therefore a solitary-wave theory on customary lines will show the length λ of the wave to increase indefinitely with h_2, since, according to the general argument explained in §1, λ is proportional to the square root of this coefficient. But this result is clearly spurious, because the binomial expansion of (2.2) in powers of k and the limit $h_2 \to \infty$ cannot correctly be taken in this order.

Proceeding correctly to take the limit first, having

$$\lim_{h_2 \to \infty} (k \coth kh_2) = |k|,$$

we obtain from (2.2) for the case of infinite depth

$$c = c_0\{1 - \tfrac{1}{2}(\rho_2/\rho_1)h_1|k| + O(k^2 h_1^2)\}. \tag{2.5}$$

This exemplifies the type of dispersion relation (1.7) and hence implies the new type of solitary wave.

While the solitary-wave equation for the infinitely deep two-fluid system (with $h_2 = \infty$, $h_1 = h$, say: see figure 7 in §5 below) is obtainable from the general results derived in the next section, it can also be found in several simpler ways unsatisfactory for the general case. We can, for instance, complete the details of the simple illustrative argument given in § 1. First, in view of the point made earlier that the upper fluid exerts only a hydrostatic pressure on waves of extreme length, a well-known result from classical shallow-water theory is taken over. According to Lamb (1932, p. 279), we have that if the interface is given by

$$y = h + f(x, t), \tag{2.6}$$

then, for a very long wave, f approximately satisfies the non-linear equation (1.4) with $C = 3c_0/2h$. We also have from (2.5) that $\gamma = \frac{1}{2}(\rho_2/\rho_1)h$, and hence we can specify the solitary-wave equation (1.12) as

$$\rho_1\left\{-2\left(\frac{c-c_0}{c_0}\right)f + \frac{3}{2}\frac{f^2}{h}\right\} = \rho_2 h \mathscr{F}\{f\}, \tag{2·7}$$

from which the specific form of the solution (1.13) follows.

3. Main analysis

3.1. *The governing equation*

We shall first treat the problem indicated by figure 3 and later, in § 4, consider how the results of the analysis can be applied to other physical systems. The fluid lies on a rigid horizontal bottom $y = 0$, and its density ρ decreases upwards to a certain height, $y = h$ in the undisturbed state, above which the fluid is homo-

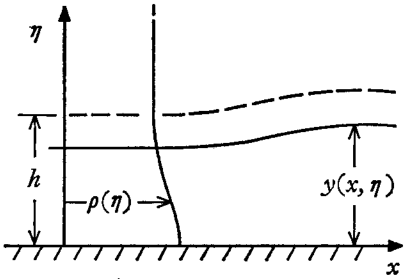

FIGURE 3. Definition sketch showing primary density distribution. The height y of the streamlines is considered as a function of their height η in the original flow and of the horizontal co-ordinate x.

geneous and extends to $y = \infty$. We shall deal only with waves of permanent form that propagate with constant velocity c towards a region at rest; accordingly, the system is viewed from a travelling frame of reference in which the motion is everywhere steady and the undisturbed fluid has a uniform velocity c in the horizontal x-direction.

As explained by Benjamin (1966, p. 248), one may take the height y of the individual streamlines in this reference frame to be the dependent variable, and let the independent variables be x and the original height η of the respective streamlines:† thus

$$y = y(x, \eta). \tag{3.1}$$

Then, on the assumption that the fluid is incompressible and non-diffusive, the density everywhere is expressible by

$$\rho = \rho(\eta), \tag{3.2}$$

† Note that $c\eta$ is the stream-function.

which represents the property that ρ is constant along each streamline. And by transformation of the equation for the stream-function (e.g. see Yih 1965, p. 76; or Benjamin 1966, equation (2.8) with (3.2) substituted) it can easily be shown that

$$\rho\left(\frac{y_{xx}}{y_\eta}-\frac{2y_x y_{x\eta}}{y_\eta^2}\right)-\frac{1}{2}\left(\frac{\rho}{y_\eta^2}\right)_\eta(1+y_x^2)=\rho_\eta\left\{\frac{g}{c^2}(y-\eta)-\tfrac{1}{2}\right\}, \tag{3.3}$$

which serves as the governing dynamical equation. Our object is to find approximate solutions of this second-order differential equation for $y(x,\eta)$, subject to the conditions $y=0$ for $\eta=0$ and $y-\eta\to 0$ for $\eta\to\infty$. These solutions are treated as perturbations of the solution $y=\eta$ which represents the undisturbed state, and they are developed by considering two regions of the flow separately: (I) the lower region $\eta\leqslant h$ for which a long-wave approximation can be made; and (II) the upper region $\eta\geqslant h$ wherein ρ is constant and therefore the right-hand side of (3.3) vanishes. At the interface $\eta=h$ between the two regions, obviously y must be a continuous function of η, and y_η also must if ρ is assumed to be continuous across the interface. The case of a two-fluid system may be included by placing the jump in density just below $\eta=h$ (see Example 1 in § 5).

We note that the horizontal and vertical components of velocity are c/y_η and cy_x/y_η, respectively (cf. Benjamin 1966, equations (3.2)), but no explicit consideration of the velocity field will be needed henceforth.

3.2. *Infinitesimal sinusoidal waves*

After substitution of

$$y=\eta+\epsilon\zeta(x,\eta) \tag{3.4}$$

in (3.3), linearization in ϵ gives

$$\rho\zeta_{xx}+(\rho\zeta_\eta)_\eta-\rho_\eta\kappa\zeta=0, \tag{3.5}$$

with $\kappa=g/c^2$; and in region II this equation reduces to

$$\zeta_{xx}+\zeta_{\eta\eta}=0. \tag{3.6}$$

Putting $\zeta(x,\eta)=\phi(\eta)\,e^{ikx}$ with k real, and requiring that ζ vanish for $\eta\to\infty$, we deduce from (3.6) that

$$\phi(\eta)=\phi(h)\,e^{-|k|(\eta-h)}\quad\text{for}\quad \eta\geqslant h. \tag{3.7}$$

And for region I we obtain from (3.5)

$$\frac{d}{d\eta}\left(\rho\frac{d\phi}{d\eta}\right)-\left(\rho k^2+\kappa\frac{d\rho}{d\eta}\right)\phi=0. \tag{3.8}$$

The boundary condition at the bottom is

$$\phi(0)=0; \tag{3.9}$$

and, by virtue of (3.7), the condition on the solution of (3.8) at the upper boundary of region I is

$$\phi_\eta(h)=-|k|\,\phi(h), \tag{3.10}$$

since both ϕ and ϕ_η are required to be continuous at $\eta=h$.

Equation (3.8) coupled with (3.9) and (3.10) constitute a Sturm–Liouville system of the standard type, determining a set of eigenvalues $\kappa^{(n)}$ $(n=1,2,3,\ldots)$ for given k. Since $\rho_\eta\leqslant 0$ (by the assumption of stability), the basic conclusions of the Sturmian theory (Ince 1926, ch. 10) tell us that $0 < \kappa^{(1)} < \kappa^{(2)} < \dots$ and that the set generally has no limit-point except $\kappa = \infty$. The respective eigenfunctions $\phi^{(n)}$ have exactly $n-1$ zeros in the open interval $(0, h)$, and so only the first, having no zero in the open interval, represents a wave wholly of elevation or depression. The first wave mode is the most important physically since $\kappa^{(1)}$ is least, and therefore the wave speed $c^{(1)}$ is greatest, for given k. On the understanding that a particular wave mode from a possibly infinite sequence of modes is always separately in question, the indices (n) are conveniently omitted henceforth.

By means of the Sturmian comparison theorems it is easy to show that, for any particular mode, c is largest at $k = 0$. But, as discussed previously, c does not have a smooth maximum in the present case, essentially because of the dependence on $|k|$ in (3.10). To investigate the properties of the system for small values of k, we may assume the existence of the power-series expansions

$$\left.\begin{aligned} \phi(\eta) &= \phi_0(\eta) + |k|\,\phi_1(\eta) + k^2\phi_2(\eta) + \dots, \\ \kappa &= \kappa_0 + \kappa_1|k| + \kappa_2 k^2 + \dots, \end{aligned}\right\} \tag{3.11}$$

and evaluate the leading coefficients successively. A normalization of the solution ϕ may be specified for convenience; this hardly matters at the stage of approximation to be reached here, but to be definite we take

$$\phi(h) = \phi_0(h) = 1, \tag{3.12}$$

implying the set of conditions $\phi_m(h) = 0$ for $m > 0$ which would be convenient at later stages.

The first step gives

$$\left.\begin{aligned} \frac{d}{d\eta}\left(\rho\frac{d\phi_0}{d\eta}\right) - \kappa_0\frac{d\rho}{d\eta}\phi_0 &= 0, \\ \phi_0 = 0 \quad \text{at} \quad \eta &= 0, \\ \frac{d\phi_0}{d\eta} = 0 \quad \text{at} \quad \eta &= h. \end{aligned}\right\} \tag{3.13}$$

Note that this Sturm–Liouville system determines the speed $c_0 = (g/\kappa_0)^{\frac{1}{2}}$ of infinitesimal waves of extreme length. The second step gives

$$\left.\begin{aligned} \frac{d}{d\eta}\left(\rho\frac{d\phi_1}{d\eta}\right) - \kappa_0\frac{d\rho}{d\eta}\phi_1 &= \kappa_1\frac{d\rho}{d\eta}\phi_0, \\ \phi_1 = 0 \quad \text{at} \quad \eta &= 0, \\ \frac{d\phi_1}{d\eta} = -\phi_0 \quad \text{at} \quad \eta &= h. \end{aligned}\right\} \tag{3.14}$$

One component of the complementary function for the differential equation (3.14) has been defined as ϕ_0. Hence it readily follows that the solution satisfying the first boundary condition and complying with (3.12) is

$$\begin{aligned} \phi_1(\eta) &= \kappa_1\phi_0(\eta)\int_h^\eta \frac{1}{\rho(\eta')\phi_0^2(\eta')}\left\{\int_0^{\eta'} \frac{d\rho(\eta'')}{d\eta''}\phi_0^2(\eta'')\,d\eta''\right\}d\eta' \\ &= -\frac{\kappa_1}{\kappa_0}\phi_0(\eta)\int_h^\eta \frac{1}{\rho(\eta')\phi_0^2(\eta')}\left\{\int_0^{\eta'} \rho(\eta'')\left[\frac{d\phi_0(\eta'')}{d\eta''}\right]^2 d\eta''\right\}d\eta', \end{aligned} \tag{3.15}$$

where the second expression follows from the first by use of the differential equation (3.13) for ϕ_0 and integration by parts. The substitution of (3.15) into the second boundary condition (3.14) then leads to

$$\kappa_1 = \kappa_0 \rho(h) \Big/ \int_0^h \rho \left(\frac{d\phi_0}{d\eta}\right)^2 d\eta, \tag{3.16}$$

which shows, as expected, that κ_1 is always positive. We note that

$$c = c_0\{1 - \tfrac{1}{2}(\kappa_1/\kappa_0)\,|k| + \ldots\}, \tag{3.17}$$

thus demonstrating the type of dispersion relation discussed generally in § 1. Evidently the value of the integral in (3.16) will increase with the number of oscillations of the particular ϕ_0 in the interval $(0, h)$, so that the κ_1 will be smaller for larger mode number n. Thus, for the higher modes the peak of wave speed at $k = 0$ will be progressively less sharp.

3.3. *Long waves of finite amplitude*

To obtain an approximation to finite stationary waves, we must proceed as in customary solitary-wave theory to develop at least two stages of an expansion in powers of the small parameter ϵ measuring amplitude. At the same time x-derivatives in the equations must be arranged in order of magnitude by specifying a suitably stretched horizontal scale. Accordingly we write, for application in region I,

$$y - \eta = \epsilon\zeta = \epsilon\zeta_{(0)}(X, \eta) + \epsilon^2\zeta_{(1)}(X, \eta) + \ldots, \tag{3.18}$$ †

and, guided by the considerations outlined in § 1, we specify

$$X = \epsilon x, \tag{3.19}$$

assuming the waves to be so long that derivatives with respect to X are of the same order of magnitude as the functions differentiated. Also, the wave speed is expanded in the form

$$c^2 = c_0^2\{\Delta_{(0)} + \epsilon\Delta_{(1)} + \ldots\}, \tag{3.20}$$

where c_0 is the speed of extremely long infinitesimal waves as considered in § 3.2.

Now, it is clear that the formal expansion (3.18), in which $\zeta_{(0)}(X, \eta)$ is supposed to be free from explicit dependence on ϵ, will serve only for region I (the layer $0 \leqslant \eta \leqslant h$), not for the infinite region II which has no length scale upon which to define a long-wave approximation. The way to deal with region II is shown by the preceding discussion of linearized theory: just as in developing the k-expansion, we must consider a first approximation akin to (3.7) for this region, in effect anticipating several stages of the expansion (3.18) for region I. To use words that have become very familiar nowadays, we might say that inner and outer expansions are required, to be matched across the interface $\eta = h$. Since the governing equation (3.3) is second-order, it is merely required, as in the linearized theory, to make both y and y_η continuous across $\eta = h$. Again, if the division into two regions is not precisely defined but instead the density approaches a constant value asymptotically, then the matching can be over a region common to I and II,

† Note the different meanings of the three forms of indices: inferior (n) as used here, superior (n) and inferior m as used previously.

following the familiar method expounded in the book by Van Dyke (1964). For simplicity the analysis will be developed with reference only to the case of a precise interface, but the extension to the latter case will be made clear by Example 3 in § 5.

In region II a first approximation to (3.3) is

$$\epsilon^2\zeta_{XX}+\zeta_{\eta\eta}=0 \quad \text{for} \quad \eta \geqslant h, \tag{3.21}$$

and the remainder from (3.3) is seen to be only $O(\epsilon^5)$ when (3.21) is satisfied. We could, therefore, proceed to $O(\epsilon^4)$ in developing the inner expansion (3.18) before needing a higher approximation for region II. Equation (3.21) is the same as (3.6), of course, but it will be solved in the present form—at least for the time being—to make clear the dependence on ϵ. Suppose now that the vertical displacement at the interface is given by

$$\zeta(X,h) = F(X), \tag{3.22}$$

where $F(X)$ is either periodic with period $2L = 2\epsilon l$, so that

$$F(X) = \sum_{N=-\infty}^{\infty} A_N \exp\left[-i\pi NX/L\right] \tag{3.23}$$

with

$$A_N = \frac{1}{2L}\int_0^{2L} F(X)\exp\left[i\pi NX/L\right]dX, \tag{3.24}$$

or is a function representing a solitary wave, so that

$$F(X) = \int_{-\infty}^{\infty} \tilde{F}(K)\exp\left[-iKX\right]dK \tag{3.25}$$

with

$$\tilde{F}(K) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(X)\exp\left[iKX\right]dX. \tag{3.26}$$

Then, respective to the two cases, the solution of (3.21) vanishing for $\eta \to \infty$ is either

$$\zeta = \sum_{-\infty}^{\infty} A_N \exp\left[-\pi\{iNX+\epsilon|N|(\eta-h)\}/L\right], \tag{3.27}$$

or

$$\zeta = \int_{-\infty}^{\infty} \tilde{F}(K)\exp\left[-\{iKX+\epsilon|K|(\eta-h)\}\right]dK. \tag{3.28}$$

Hence we have

$$\zeta_\eta(X,h) = -\epsilon\mathscr{F}\{F(X)\}, \tag{3.29}$$

where $\mathscr{F}\{F(X)\}$ is defined either by

$$\mathscr{F}\{F(X)\} = \frac{\pi}{L}\sum_{-\infty}^{\infty}|N|A_N\exp\left[-i\pi NX/L\right], \tag{3.30}$$

or, as considered already in § 1,

$$\mathscr{F}\{F(X)\} = \int_{-\infty}^{\infty}|K|\tilde{F}(K)\exp\left[-iKX\right]dK. \tag{3.31}$$

(Another representation of the operator $\mathscr{F}$ will be discussed in the appendix to this paper.) The pair of equations (3.22) and (3.29) amount to a generalization of (3.10) and may similarly be regarded as an upper boundary condition on region I, valid to $O(\epsilon^4)$.

After the expansion (3.18) has been substituted into (3.3), the separation of terms $O(\epsilon)$ gives

$$\frac{\partial}{\partial\eta}\left(\rho\frac{\partial\zeta_{(0)}}{\partial\eta}\right)-\frac{\kappa_0}{\Delta_{(0)}}\frac{d\rho}{d\eta}\zeta_{(0)}=0, \tag{3.32}$$

and the respective boundary conditions are

$$\zeta_{(0)}(X,0)=0,\quad \zeta_{(0)\eta}(X,h)=0, \tag{3.33}$$

the latter of which follows when (3.18) is substituted into (3.29). Hence we deduce that, to this first approximation,

$$\Delta_{(0)}=1,\quad \zeta_{(0)}=\frac{F(X)\phi_0(\eta)}{\phi_0(h)}, \tag{3.34}$$

where ϕ_0 is the eigenfunction determined by (3.13) and $F(X)$ is an arbitrary function. To save a lot of writing subsequently, we can again, as in (3.12), arbitrarily make

$$\phi_0(h)=1. \tag{3.35}$$

To complete a normalization corresponding to (3.12), we can further choose to treat $\epsilon F(X)$ as the final solution for the displacement at $\eta=h$, so specifying that $\zeta_{(1)},\zeta_{(2)},\zeta_{(3)},\ldots$ should all be made zero at $\eta=h$. The subsequent stages of approximation will then present a succession of equations for $F(X)$, each one determining this solution with greater precision than at the previous stage.

The second approximation to (3.3) in region I, when terms $O(\epsilon^2)$ are collected, gives

$$\begin{aligned}\frac{\partial}{\partial\eta}\left(\rho\frac{\partial\zeta_{(1)}}{\partial\eta}\right)-\kappa_0\frac{d\rho}{d\eta}\zeta_{(1)}&=-\Delta_{(1)}\frac{\partial}{\partial\eta}\left(\rho\frac{\partial\zeta_{(0)}}{\partial\eta}\right)+\frac{3}{2}\frac{\partial}{\partial\eta}\left[\rho\left(\frac{\partial\zeta_{(0)}}{\partial\eta}\right)^2\right]\\&=-\Delta_{(1)}\frac{d}{d\eta}\left(\rho\frac{d\phi_0}{d\eta}\right)F+\frac{3}{2}\frac{d}{d\eta}\left[\rho\left(\frac{d\phi_0}{d\eta}\right)^2\right]F^2,\end{aligned} \tag{3.36}$$

and the boundary conditions are

$$\zeta_{(1)}(X,0)=0,\quad \zeta_{(1)\eta}(X,h)=-\mathscr{F}\{F(X)\}, \tag{3.37}$$

the latter of which comes from (3.29). Equation (3.36) is soluble in exactly the same way as the equation (3.14) that was presented at the second stage of deriving the k-expansion for infinitesimal waves. Denoting the right-hand side of (3.36) by $R(X,\eta)$ for short, we find that the solution satisfying the first of the boundary conditions (3.37) and complying with the normalization explained below (3.35) is

$$\zeta_{(1)}=\phi_0(\eta)\int_h^\eta\frac{1}{\rho(\eta')\phi_0^2(\eta')}\left\{\int_0^{\eta'}R(X,\eta'')\phi_0(\eta'')\,d\eta''\right\}d\eta'. \tag{3.38}$$

Hence, recalling the last of (3.13) and also (3.35), we obtain from the second boundary condition

$$\frac{1}{\rho(h)}\int_0^h R\phi_0\,d\eta=-\mathscr{F}\{F(X)\}. \tag{3.39}$$

When R is written in full here and an integration by parts is made, the integrated terms are seen to vanish in consequence of the boundary conditions (3.13) on ϕ_0, and thus it follows that

$$-\frac{\Delta_{(1)}F}{\rho(h)}\int_0^h\rho\left(\frac{d\phi_0}{d\eta}\right)^2d\eta+\frac{3F^2}{2\rho(h)}\int_0^h\rho\left(\frac{d\phi_0}{d\eta}\right)^3d\eta=\mathscr{F}\{F(X)\}. \tag{3.40}$$

This equation for $F(X)$ has the same form as (1.12) and it is the central result of this analysis. After solving (3.40) we could go on to substitute the expression (3.38) for $\zeta_{(1)}$ into (3.18) and so obtain an approximation for $y-\eta$ explicitly to $O(\epsilon^2)$. But this step would add nothing of qualitative significance further to what we already have. The essential character of the solution for long waves of finite amplitude and permanent form appears to be unfolded in the expression (3.34) for $\zeta_{(0)}$, when the second stage of approximation is used to determine $F(X)$ as shown here. This situation is familiar from classical solitary-wave theory, and reference may be made to Benjamin (1967, appendix) for a discussion of the point in a context rather similar to the present one.

[Note that since ϕ_0 varies between zero and an extremum over the interval $(0, h)$, there appears to be no likelihood of the second integral in (3.40) having a (non-dimensionalized) value much less than $O(1)$—as it well might if, for instance, ϕ_0 were to oscillate like $\sin(n\pi\eta/h)$ between zeros at both limits of integration. Thus, *relative* to the second, non-linear term in (3.40), the implied error in the present approximation is simply $O(\epsilon)$, and therefore the assumption of small amplitude (compared with h) is sufficient justification for the approximation. This point is noteworthy because the mentioned difficulty does in fact arise in the treatment, along lines similar to the present, of long internal waves in fluids of limited total depth. For that case in general, it was shown by Benjamin (1966, pp. 254, 255) that the coefficient of the leading non-linear term (in effect, the counterpart of the coefficient of F^2 in (3.40)—i.e. V as defined by (3.53) below) is only $O(b)$, where b is the fractional density variation. Moreover, when η is used as an independent variable like it is here, the error in an equation corresponding to (3.40) is $O(\epsilon)$ and therefore is $O(\epsilon/b)$ relative to the retained non-linear term. Consequently, the approximation implies a very severe limitation on amplitude ϵ if, as is usual, b is rather small. The difficulty was shown to be obviated by replacing $\phi_0(\eta)$ with $\phi_0(y)$ in the approximate solution corresponding to (3.34): this adjustment reduces the relative error to $O(\epsilon)$, independently of b. As already explained, however, the present solution is free from the difficulty in question, and so apparently no further precision would be gained by such an adjustment.]

3.4. *Summary of preceding results*

It will be helpful to sum up the main equations and recast them in the forms most convenient for application to specific problems. The parameter ϵ has now served its purpose in the systematic derivation of approximate equations and we can dispense with it henceforth, rewriting the equations directly in terms of physical variables and letting the smallness of perturbations be implicit. A horizontal length scale $\lambda = O(\epsilon^{-1}h) \equiv O(a^{-1}h^2)$ will then emerge automatically in the results. For instance, we put

$$\epsilon\Delta_{(1)} \equiv \frac{c^2-c_0^2}{c_0^2}, \tag{3.41}$$

and we present the solution of the problem as an expression for

$$y-\eta = z(x,\eta), \tag{3.42}$$

identifying this with the approximation $\epsilon\zeta_{(0)}(X,\eta)$.

Considered in this way, the results of the preceding subsection amount to the following. First, the vertical displacement of the interface $\eta = h$ is expressed by

$$z(x,h) = f(x), \tag{3.43}$$

where either, for periodic waves,

$$f(x) = \sum_{N=-\infty}^{\infty} A_N \exp[-i\pi Nx/l] \tag{3.44}$$

with

$$A_N = \frac{1}{2l}\int_0^{2l} f(x)\exp[i\pi Nx/l]\,dx, \tag{3.45}$$

or, for solitary waves,

$$f(x) = \int_{-\infty}^{\infty} \tilde{f}(k)\exp[-ikx]\,dk \tag{3.46}$$

with

$$\tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\exp[ikx]\,dx. \tag{3.47}$$

Then the solution in region I is

$$z = f(x)\,\phi_0(\eta), \tag{3.48}$$

where ϕ_0 is defined by the Sturm–Liouville system (3.13),† which also defines c_0, and by the normalization (3.35). And the solution in region II is either, for periodic waves,

$$z = \sum_{-\infty}^{\infty} A_N \exp[-\pi\{iNx + |N|\,(\eta-h)\}/l], \tag{3.49}$$

or, for solitary waves,

$$z = \int_{-\infty}^{\infty} \tilde{f}(k)\exp[-\{ikx + |k|\,(\eta-h)\}]\,dk. \tag{3.50}$$

Finally, by (3.40), the function $f(x)$ must satisfy

$$Uf - Vf^2 + \mathscr{F}\{f\} = 0, \tag{3.51}$$

where

$$U = \left(\frac{c^2-c_0^2}{c_0^2}\right)\frac{1}{\rho(h)}\int_0^h \rho\left(\frac{d\phi_0}{d\eta}\right)^2 d\eta, \tag{3.52}$$

$$V = \frac{3}{2\rho(h)}\int_0^h \rho\left(\frac{d\phi_0}{d\eta}\right)^3 d\eta, \tag{3.53}$$

and either, for periodic waves,

$$\mathscr{F}\{f\} = \frac{\pi}{l}\sum_{-\infty}^{\infty} A_N|N|\exp[-i\pi Nx/l], \tag{3.54}$$

or, for solitary waves,

$$\mathscr{F}\{f\} = \int_{-\infty}^{\infty} |k|\tilde{f}(k)\exp[-ikx]\,dk. \tag{3.55}$$

3.5. *Periodic waves*

To find a periodic solution of (3.51), Fourier-series expansions of each of the three functions f, f^2 and $\mathscr{F}\{f\}$ appearing in the equation may be supposed to exist. A solution is hence established if it can be shown that the equation is satisfied

† It is perhaps worth a reminder here that any one of the set of wave modes determined by (3.13) is admissible: i.e. $\phi_0 \equiv \phi_0^{(n)}$, $c_0 \equiv c_0^{(n)}$.

formally by such a set of expansions taken term by term, and that the series for f is convergent. Accordingly we take

$$f^2 = \sum_{N=-\infty}^{\infty} B_N \exp[-i\pi Nx/l], \tag{3.56}$$

so that $$B_N = \frac{1}{2l}\int_0^{2l}\left\{\sum_{M=-\infty}^{\infty} A_M \exp[-i\pi Mx/l]\right\}^2 \exp[i\pi Nx/l]\,dx$$

$$= \sum_{M=-\infty}^{\infty} A_M A_{N-M}. \tag{3.57}$$

Then (3.51) gives

$$\left\{U+\frac{\pi}{l}|N|\right\}A_N = V\sum_{M=-\infty}^{\infty} A_M A_{N-M}, \tag{3.58}$$

which needs to be satisfied for all N.

Now, let us try

$$A_N = \Delta\exp[-p|N|], \tag{3.59}$$

where Δ and p are real constants, with $p > 0$ necessarily to make the series (3.44) convergent. We have

$$\sum_{M=-\infty}^{\infty} A_M A_{N-M} = \Delta^2 \sum_{M=-\infty}^{\infty} \exp[-p(|M|+|N-M|)]$$

$$= \Delta^2\exp[-p|N|]\left\{|N|+1+2\sum_{s=1}^{\infty}\exp[-2ps]\right\}$$

$$= \Delta^2\exp[-p|N|]\{|N|+\coth p\}. \tag{3.60}$$

Hence (3.56) reduces to

$$U+\frac{\pi}{l}|N| = V\Delta\{|N|+\coth p\}, \tag{3.61}$$

which is satisfied for all N if

$$\Delta = \frac{U}{V}\tanh p, \tag{3.62}$$

and $$\tanh p = \frac{\pi}{lU}. \tag{3.63}$$

Thus, with Δ and p given by (3.62) and (3.63), an exact solution of (3.51) is

$$f(x) = \sum_{-\infty}^{\infty} \Delta\exp[-(p|N|+i\pi Nx/l)] \tag{3.64}$$

$$= \Delta\,\mathrm{Re}\left\{1+2\sum_{1}^{\infty}\exp[-(p+i\pi x/l)N]\right\} = \Delta\,\mathrm{Re}\coth\{\tfrac{1}{2}(p+i\pi x/l)\},$$

from which we obtain finally

$$f(x) = \frac{\frac{1}{2}\Delta\sinh p}{\cosh^2(\frac{1}{2}p)-\cos^2(\frac{1}{2}\pi x/l)}. \tag{3.65}$$

To begin the interpretation of this solution, we first note from (3.63) that U must be positive. Therefore the solution as here presented applies only to the case of 'supercritical' speeds ($c^2 > c_0^2$), although a simple modification will later be shown to cover the subcritical case. It is also noted from (3.62) that Δ takes the sign of V. The sign is always positive for the first wave mode, because then ϕ_0 has no zero in the open interval $(0, h)$ and consequently, in the definition (3.53) of V, $d\phi_0/d\eta > 0$ throughout the interval. But nothing definite can be said in general about the sign of V for the higher modes.

To be specific, therefore, let us refer to the first mode for the discussion of wave properties. As is obvious from (3.64), the mean value of $f(x)$ is given by

$$\bar{f} = A_0 = \Delta; \tag{3.66}$$

and so, for supercritical wave trains in the first mode, there is a mean upward displacement of the surface $\eta = h$ from its undisturbed level.† According to (3.65), the elevation of the wave crests is given by

$$f_{\max} = \frac{\frac{1}{2}\Delta \sinh p}{\sinh^2(\frac{1}{2}p)} = \Delta \coth(\tfrac{1}{2}p), \tag{3.67}$$

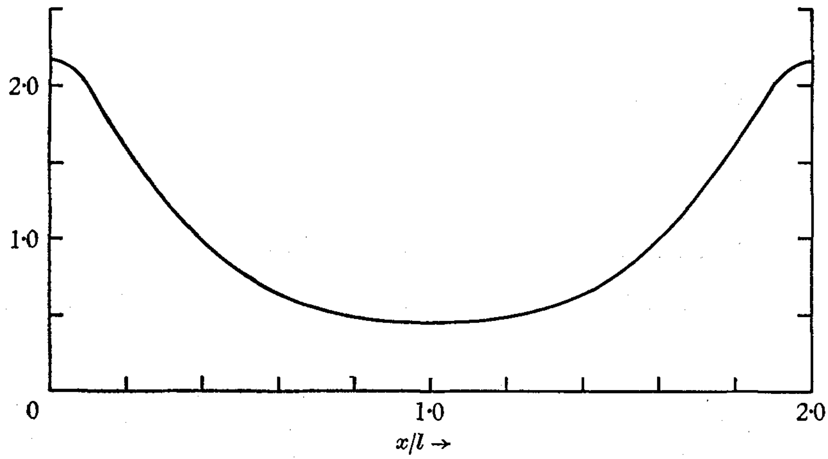

FIGURE 4. Graph of the function $f(x) = \frac{1}{2}\sinh p/\{\cosh^2(\frac{1}{2}p) - \cos^2(\frac{1}{2}\pi x/l)\}$ with $p = 1{\cdot}0$, illustrating form of periodic wave over one period. The amplitude is 1·720, and the mean value of the function is 1·0.

and the elevation of the troughs by

$$f_{\min} = \frac{\frac{1}{2}\Delta \sinh p}{\cosh^2(\frac{1}{2}p)} = \Delta \tanh(\tfrac{1}{2}p). \tag{3.68}$$

Hence the wave amplitude may be defined as

$$\hat{f} = f_{\max} - f_{\min} = 2\Delta \operatorname{cosech} p. \tag{3.69}$$

Also, the elevation midway between crests and troughs is

$$\tfrac{1}{2}(f_{\max} + f_{\min}) = \Delta \coth p = U/V, \tag{3.70}$$

which exceeds $\bar{f}$ in the ratio $(\coth p):1$, indicating that the waves are peaked upwards—i.e. sharper at the crests than at the troughs. An example of the wave-form, with $p = 1{\cdot}0$, is shown in figure 4.

† By an obviously inherent property of the theoretical model, there is also a mean displacement $z = \Delta$ of all surfaces $\eta = \text{const.} \geqslant h$ in region II (see equation (3.75) below).

The outstanding practical application for the present form of solution is to the wave train developed by a weak bore. The idea is represented in figure 5. A bore will advance towards the undisturbed fluid at a supercritical speed (i.e. faster than any infinitesimal wave) and will generally evolve in the first mode since the respective speed of propagation is greatest. As indicated by (3.66), the bore will

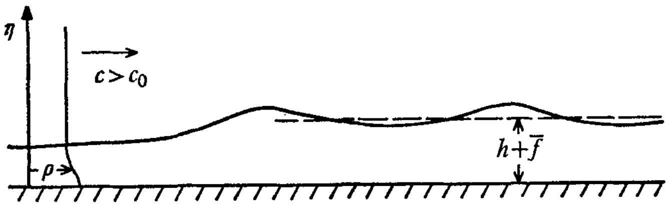

FIGURE 5. Illustration of undular bore. Speed is supercritical, and mean level of heterogeneous region I is raised.

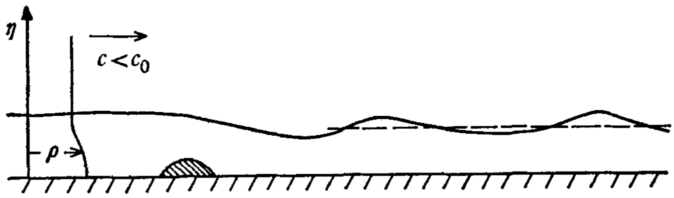

FIGURE 6. Illustration of subcritical wave train formed behind solid obstacle. Mean level of heterogeneous region I is lowered.

therefore be characterized by an increase in the mean level of the inner region I, thus in an obvious sense resembling an open-channel bore. With regard to heterogeneous-fluid systems of limited total depth, the application of cnoidal-wave solutions to the description of internal bores was discussed at some length by Benjamin (1966, §§3.7, 3.8), who showed that the amount of dissipation occurring at the front of a bore fixes the properties of the trailing wave-train. Unfortunately, since the present analysis has proceeded on different lines, we are not now in a position to evaluate the effects of dissipation; but the following general interpretation is suggested by the previous work.

If the bore is very weak, the loss of energy at its front is likely to be only a small fraction of the maximum amount possible at a given propagation speed (i.e. at a given U as defined by (3.52)). Then the waves formed are describable by letting p take extremely small values, so that $\Delta = (U/V)\tanh p$ is also very small. In this case the period of the waves, $2l = 2\pi/\Delta V$, is very great, and the wave-form determined by (3.65) is markedly unsymmetrical about the mean level, being greatly stretched at the troughs. The wave-train then resembles a succession of solitary waves (see § 3.6 below). In stronger bores a larger fraction of the possible energy loss may occur, in which case the waves formed are describable by taking larger values of p. And the extreme case where the maximum loss occurs is represented by the limit $p \to \infty$. Then the wave amplitude $\hat{f}$ shrinks to zero and the wave-form becomes sinusoidal. Also, equations (3.62) and (3.65) show that

$\bar{f} \to U/V$, and (3.63) shows that $l \to \pi/U$. Thus, in this limit the bore comprises a transition to a uniform state, with $f = \bar{f} = U/V$, which in a frame of reference moving with the bore front is subcritical (since we have just seen that infinitesimal waves with the speed c and with finite wavelength $2l = 2\pi/U$ can occur upon it. All the properties noted in this interpretation correspond to properties of surface bores in open channels, and of internal bores in heterogeneous fluids of finite depth (cf. Benjamin & Lighthill 1954; Benjamin 1966).

It remains to find a solution for waves of finite amplitude at subcritical speeds (i.e. with $c^2 < c_0^2$ and therefore $U < 0$). This category includes, of course, waves formed in the wake of a solid body moving horizontally at a steady speed less than c_0 (see figure 6); and the required solution should, in the limit of small amplitude, recover the results of the linearized theory presented in § 3.2. First we observe that the substitution of

$$f(x) = \frac{U}{V} + \xi(x) \tag{3.71}$$

into (3.51) gives

$$-U\xi - V\xi^2 + \mathscr{F}\{\xi\} = 0. \tag{3.72}$$

Thus, when U is negative, ξ satisfies the same equation as f does in the case of positive U; therefore ξ is given by the previous solution (3.65) with $|U|$ replacing U in the specifications (3.62) and (3.63) of Δ and p. It follows at once that in the subcritical case the mean value of f is given by

$$\bar{f} = -\frac{|U|}{V}(1 - \tanh p). \tag{3.73}$$

This shows that the formation of waves in the first mode (i.e. with $V > 0$), behind a body moving at subcritical speed, will lower the mean level of region I.

As a guide to the practical application of the modified solution (3.71), the parameter p may be reckoned to depend inversely on wave resistance, even though, as with regard to dissipation previously, we are not in a position to specify its precise relation to this physical property. The case when p is extremely small corresponds to a maximum realizable wave resistance for a given $c^2 < c_0^2$. Then the waves have a maximum amplitude $2|U|/V$ and an exaggerated period, so that they resemble a succession of solitary waves just as in the former, supercritical case when p was made extremely small—except that now they are based on a mean level $\bar{f} = -|U|/V$. The uniform state $f = -|U|/V$ is therefore supercritical in a reference frame moving with the (subcritical) velocity c implied by U. The other extreme, $p \to \infty$, again corresponds to infinitesimal waves. Equation (3.73) gives $\bar{f} = 0$ in the limit, and the modified version of (3.63) gives

$$-U = \pi/l. \tag{3.74}$$

Noting that π/l is the same as wave-number $|k|$, we readily confirm that this is equivalent to the result (3.16), or (3.17), obtained for infinitesimal waves in §3.2.

Finally, the specific form of the periodic solution (3.49) is noted, which describes the displacement $z(x, \eta)$ throughout region II. Substituting (3.59), we have

$$z = \Delta \sum_{N=-\infty}^{\infty} \exp\left[-\left[|N|\{p + \pi(\eta - h)/l\} + i\pi N x/l\right]\right]. \tag{3.75}$$

Hence the same steps that took us from (3.64) to (3.65) lead to

$$z = \frac{\frac{1}{2}\Delta \sinh\{p + \pi(\eta - h)/l\}}{\cosh^2\{\frac{1}{2}p + \frac{1}{2}\pi(\eta - h)/l\} - \cos^2(\frac{1}{2}\pi x/l)}. \tag{3.76}$$

This is the result for the case of supercritical speeds, and the corresponding results for the other case should be obvious from the preceding two paragraphs.

3.6. *Solitary waves*

The solitary-wave solution may be obtained tentatively from the results of § 3.5 by taking the limit $l \to \infty$, $p \to 0$, $\Delta \to 0$ in such a way that

$$lp \to \pi\lambda, \quad l\Delta \to \tfrac{1}{2}\pi a\lambda, \tag{3.77}$$

where λ and a are finite constants, λ being necessarily positive and a having the sign of V. Equation (3.65) gives in the limit

$$f(x) = \frac{a\lambda^2}{x^2 + \lambda^2}, \tag{3.78}$$

while (3.62) and (3.63) give

$$a = 2U/V, \quad \lambda = 1/U. \tag{3.79}$$

The same procedure applied to the periodic solution (3.71) for the subcritical case merely gives a wave of the same form superposed on the uniform supercritical state represented by $f = U/V$ ($U < 0$). Hence we conclude that the present solitary waves, like ones of the more familiar kind, always have supercritical speeds.

It is a simple matter to confirm the solution (3.78) directly by substitution in equations (3.51) and (3.55). There is some interest, however, in considering the Fourier transform of equation (3.51), thus proceeding in a way analogous to that taken in § 3.5 to prove the periodic solution. The transform of f^2 can be expressed by a convolution integral, whereupon (3.51) gives

$$(U + |k|)\bar{f}(k) = V\int_{-\infty}^{\infty} \bar{f}(k')\bar{f}(k-k')\,dk', \tag{3.80}$$

in which $\bar{f}$ is defined by (3.47). Now, let us try

$$\bar{f}(k) = \tfrac{1}{2}a\lambda \exp[-\lambda|k|]. \tag{3.81}$$

We have

$$\begin{aligned}\int_{-\infty}^{\infty} \bar{f}(k')\bar{f}(k-k')\,dk' &= \tfrac{1}{4}a^2\lambda^2 \int_{-\infty}^{\infty} \exp[-\lambda(|k'| + |k-k'|)]\,dk' \\ &= \tfrac{1}{4}a^2\lambda^2 \exp[-\lambda|k|]\{\lambda^{-1} + |k|\},\end{aligned} \tag{3.82}$$

and hence see that (3.80) is satisfied when a and λ are as specified by (3.79). It follows by (3.46) that the solution of (3.51) is

$$\begin{aligned}f(x) &= \tfrac{1}{2}a\lambda \int_{-\infty}^{\infty} \exp[-(\lambda|k| + ikx)]\,dk \\ &= \frac{a\lambda^2}{x^2+\lambda^2},\end{aligned} \tag{3.78$'$}$$

as found before.

Substituting (3.81) into (3.50), we find that the displacement $z(x, \eta)$ in region II is given by

$$z = \frac{a(\lambda+\eta-h)^2}{x^2+(\lambda+\eta-h)^2}. \tag{3.83}$$

This result shows that the displacement for $\eta \geqslant h$ is the same as if the homogeneous fluid were extended downwards to $\eta = -\infty$ and a dipole were placed at $x = 0$, $\eta = h-\lambda$. Note that since $\lambda/h = O(\epsilon^{-1}) \gg 1$ by hypothesis, the virtual dipole is far below the bottom $\eta = 0$.

4. Other applications

4.1. *The heterogeneous layer uppermost*

We now consider how the results of § 3 can be adapted to the other physical models shown in figure 1. The simplest case is the one shown in figure 1 (*b*), where the fluid is bounded at the top by a rigid plane and extends downwards to infinite depths. If η and y are measured downwards from the boundary, g must be replaced by $-g$ in all the equations; but we now have $d\rho/d\eta \geqslant 0$ as the condition of stability, and so the analytical problem takes precisely the same form as before. In particular, the coefficients U and V determining wave properties are given by the same formulae, (3.52) and (3.53). The conclusion that $y-\eta \geqslant 0$ everywhere for solitary waves in the first mode still holds, but now, of course, it means that the displacements are downwards.

4.2. *Effects of a free surface*

When the fluid is bounded at the top by a free surface (figure 1 (*c*)), a condition of constant pressure must be considered to replace the condition of zero vertical displacement that has so far been applied. Evaluation of Bernoulli's equation in both the original and disturbed flows gives, if y and η are measured downwards,

$$c^2\left\{\left(\frac{1+y_x^2}{y_\eta^2}\right)-1\right\} = 2gy \quad \text{at} \quad \eta = 0 \tag{4.1}$$

(cf. Benjamin 1966, equation (3.7)). And, with (3.4) substituted, the linearized version of (4.1) is

$$c^2\zeta_\eta = -g\zeta \quad \text{at} \quad \eta = 0. \tag{4.2}$$

We should at this point note that

$$\zeta = \exp[-|k|\eta - ikx] \tag{4.3}$$

satisfies the linearized differential equation (3.5) everywhere in the fluid, and also satisfies the boundary condition (4.2), provided

$$c^2 = g/|k|. \tag{4.4}$$

Thus it appears that, whatever the form of $\rho(\eta)$, infinitesimal waves are possible having the same speed as waves in a homogeneous liquid of infinite depth. This fact was observed by Lamb (1932, p. 376) and has been discussed further by Yih (1965, p. 54). Clearly, however, these are not true internal waves: rather they are surface waves which, it so happens, retain all their properties when density is allowed to vary. Their speed increases without limit as $|k|$ is reduced, rather than rising to a definite maximum like an internal-wave speed, and so there is obviously no solitary wave in this mode. The present analysis is incapable of estimating the effects of finite amplitude on these waves, and nothing more need to said about them here.

For internal waves in liquids with a free surface, (4.2) shows the first boundary condition on $\phi_0(\eta)$ to be

$$\phi_0 = -\frac{1}{\kappa_0}\frac{d\phi_0}{d\eta} \quad \text{at} \quad \eta = 0, \tag{4.5}$$

where $\kappa_0 = g/c_0^2$. The other boundary condition, at $\eta = h$, and the differential equation for ϕ_0 remain the same as in (3.13), so that again ϕ_0 and κ_0 are determined by a Sturm–Liouville system—together with the normalization (3.35). Since again ϕ_0 must rise (after $n-1$ oscillations) to a maximum of 1 at $\eta = h$, the derivative in (4.5) is $O(nh^{-1})$, where $n = 1, 2, 3, \ldots$ is the mode number. Also, from (3.13), we deduce that $\kappa_0^{-1} = O(n^{-2}h\delta\rho/\rho)$, if $\delta\rho$ is the total density variation. The right-hand side of (4.5) is therefore $O(n^{-1}\delta\rho/\rho)$, which may well be negligible when the fractional variation $\delta\rho/\rho$ is very small—as it usually is in practical applications. Thus, without much error one might take $\phi_0(0) = 0$ as the upper boundary condition, so treating the problem as if the surface were rigid. That there will be no important effect on the non-linear theory can be seen from the facts that the integral in the formula (3.52) for U is $O(n^2/h)$ and V as given by (3.53) is $O(n^3/h^2)$, showing that neither could be affected vitally by corrections of the order in question. [Note that the situation respecting long internal waves of finite amplitude in fluids of limited depth is quite different. It has been shown by Benjamin (1966, §4, Example 3) that the wave properties can change radically according to whether the upper boundary is fixed or free, irrespective of how small $\delta\rho/\rho$ is. The reason essentially is that a coefficient corresponding to V (K in the paper cited) is only $O(h^{-2}\delta\rho/\rho)$ in the case of limited total depth h.]

Nevertheless, the effects of a free surface can readily be included in the theory without further approximations, and the following summary of the argument seems worth giving. After substitution of the expansions (3.18) and (3.20), the boundary conditions are again satisfied to $O(\epsilon)$ by $\zeta_{(0)} = F(X)\,\phi_0(\eta)$, provided (4.5) is used in the definition of ϕ_0. But, to $O(\epsilon^2)$, (4.1) requires that

$$\begin{aligned}\frac{\partial\zeta_{(1)}}{\partial\eta} + \kappa_0\zeta_{(1)} &= -\Delta_{(1)}\frac{\partial\zeta_{(0)}}{\partial\eta} + \frac{3}{2}\left(\frac{\partial\zeta_{(0)}}{\partial\eta}\right)^2 \\ &= -\Delta_{(1)}\left(\frac{d\phi_0}{d\eta}\right)F + \frac{3}{2}\left(\frac{d\phi_0}{d\eta}\right)^2 F^2 \\ &= J(X), \quad \text{say, at} \quad \eta = 0. \end{aligned}\tag{4.6}$$

The expression (3.38) is evidently insufficient to satisfy (4.6), but we may add to it any term in the form $G(X)\,\psi(\eta)$, where

$$\psi = \phi_0 \int_h^\eta \frac{d\eta}{\rho\phi_0^2} \tag{4.7}$$

is the second part of the complementary function for the differential equation (3.36) [i.e. the second solution of (3.32) or of the differential equation in (3.13)] and is specified to comply with the normalization condition $\zeta_{(1)}(X,h) = 0$. Therefore we can take

$$\zeta_{(1)} = \phi_0(\eta)\int_h^{\eta}\frac{1}{\rho\phi_0^2}\left\{\int_0^{\eta'} R\phi_0\,d\eta''\right\}d\eta' + \frac{J\psi(\eta)}{\psi_\eta(0)+\kappa_0\psi(0)}, \tag{4.8}$$

which is easily seen to satisfy (4.6).

For substitution in the second boundary condition (3.37), at $\eta = h$, there follows

$$\zeta_{(1)\eta}(X,h) = \frac{1}{\rho(h)}\int_0^h R\phi_0\,d\eta + \frac{J\psi_\eta(h)}{\psi_\eta(0)+\kappa_0\psi(0)}. \tag{4.9}$$

But we have
$$\psi_\eta(0)+\kappa_0\psi(0) = \frac{1}{\rho(0)\,\phi_0(0)} \tag{4.10}$$

in consequence of (4.5), and

$$\psi_\eta(h) = \frac{1}{\rho(h)\,\phi_0(h)} = \frac{1}{\rho(h)} \tag{4.11}$$

since $\phi_{0\eta}(h) = 0$. Hence

$$\zeta_{(1)\eta}(X,h) = \frac{1}{\rho(h)}\left\{\int_0^h R\phi_0\,d\eta + J\rho(0)\,\phi_0(0)\right\}. \tag{4.12}$$

Recalling that R denotes the right-hand side of (3.36), we now integrate by parts and find that one set of integrated terms vanishes because $\phi_{0\eta}(h) = 0$, while the other cancels with the term that has J as a factor. Thus we obtain finally

$$-\zeta_{(1)\eta}(X,h) = -\frac{\Delta_{(1)}F}{\rho(h)}\int_0^h \rho\left(\frac{d\phi_0}{d\eta}\right)^2 d\eta + \frac{3F^2}{2\rho(h)}\int_0^h \rho\left(\frac{d\phi_0}{d\eta}\right)^3 d\eta, \tag{4.13}$$

which is precisely the same expression that appeared on the left-hand side of (3.40).

This result means that the formulae (3.52) and (3.53) for U and V apply equally well to the present case as to the case of a rigid boundary. It seems very remarkable that the non-linear effects of the free surface are represented in these compact formulae simply by the changes in ϕ_0 as determined by linearized theory; but, of course, the new term introduced in (4.8) affects the second-order displacements *inside* region I. A corresponding result was found for fluids of limited depth by Benjamin (1966, §3.5), using a quite different method, and this can easily be confirmed by extending the present argument to the case of a rigid boundary at $\eta = h$.

4.3. *Heterogeneous layers remote from any boundary*

We turn now to the class of problems indicated by figure 1(*d*). Here the fluid extends to great distances both above and below the layer across which the density varies. This case can be treated by a fairly obvious modification of the method developed in §3; for instance, in the Sturm–Liouville system determining ϕ_0 and c_0, the two end conditions would be that $d\phi_0/d\eta$ should vanish at both the upper and lower bounds of region I. However, a fully generalized treatment presents certain complications which do not seem worth our attention now. With regard to practical applications it seems adequate merely to note the following very simple means of using the results already obtained.

Suppose that the density variation is both small and antisymmetrical about a central plane, say $\eta = 0$, so that $\rho(\eta)+\rho(-\eta) = 2\rho(0)$ for all η. Then the gravitational forces on the fluid are antisymmetric; and on the basis of the familiar *Boussinesq approximation*, which takes the effect of the small density changes on the inertia of the fluid to be negligible, the space below the plane of symmetry may be regarded as dynamically similar to the space above. The motions in each space are thus approximately the same as if a rigid boundary were placed at $\eta = 0$, and so they can be described by direct application of the solutions given in § 3. Clearly, for solitary waves in the first mode, the displacements are upwards above $\eta = 0$ and are downwards below.

The admissibility of the Boussinesq approximation is shown by the same considerations that showed the effects of a free surface to be insignificant when $O(\delta\rho/\rho) \ll 1$. Namely, neither $U/(c^2-c_0^2)$ nor V is small enough to depend crucially on the effects neglected. Again the situation is simpler than for solitary waves in fluids of limited depth. Long (1965) and Benjamin (1966, see particularly the appendix) have shown that the approximation generally leads to serious errors in that case.

Consistent with this use of the Boussinesq approximation, the results from § 3 may be simplified somewhat. We may put $\rho = \rho(0)$ in the first term of the differential equation (3.13) for ϕ_0, thus getting

$$\frac{d^2\phi_0}{d\eta^2}-\frac{g}{c_0^2\rho(0)}\frac{d\rho}{d\eta}\phi_0 = 0, \tag{4.14}$$

and the same approximation may be made in the formulae (3.52) and (3.53) for U and V. Note that for the type of density distribution assumed, the coefficient of ϕ_0 in (4.14) is an even function of η; therefore ϕ_0 is either an even or an odd function. The case where ϕ_0 is odd is, of course, what we have in view for the present application of the theory, and no attempt will be made to deal with the even wave modes—for which, it turns out, the Boussinesq approximation is invalid as regards finite-amplitude effects.

5. Examples

The main conclusions of this paper have been reached without the need arising to specify the density distribution, other than that it belongs to the general category explained at the beginning. But now three examples will be worked out to illustrate the use of the results obtained in §§ 3 and 4. Each example requires us merely to find ϕ_0 from the linearized long-wave equations, at the same time finding c_0, and to evaluate the definite integrals (3.52) and (3.53) expressing U and V. Only the properties of solitary waves in each system will be noted explicitly, but the description of periodic long waves of finite amplitude is then obvious from the discussion in § 3.5.

Example 1. *Two-fluid systems*

First we confirm that the results of the general analysis bear out the simple deduction made at the end of § 2, where the system shown in figure 7 was discussed. To preserve the argument in the form set out in § 3, the interface between the two immiscible fluids may be considered to lie an infinitesimal distance below the level $\eta = h$ at which the boundary condition on ϕ_0 is to be applied. Below the interface, the differential equation (3.13) reduces to

$$\frac{d^2\phi_0}{d\eta^2} = 0; \tag{5.1}$$

and since ϕ_0 must be continuous everywhere (as required for continuity of the vertical displacements of the fluid), the solution vanishing on the bottom and satisfying (3.35) is

$$\phi_0 = \eta/h. \tag{5.2}$$

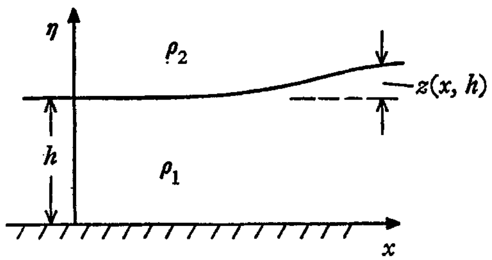

FIGURE 7. Illustration of two-fluid system in which the upper fluid extends to $\eta = \infty$.

Now, the differential equation (3.13) implies that if ϕ_0 is continuous, then $\rho c_0^2(d\phi_0/d\eta) - g\rho\phi_0$ also must be continuous, and on examination this requirement is seen to ensure the continuity of pressure (cf. Benjamin 1966, p. 263). Since $d\phi_0/d\eta = 0$ at $\eta = h$ immediately above the interface, it follows at once that

$$c_0^2 = \frac{(\rho_1 - \rho_2)gh}{\rho_1}, \tag{5.3}$$

where ρ_1 is the density of the lower fluid and ρ_2 that of the upper. This expression for c_0 checks with the results due to Lamb that were quoted in §2.

Substituting (5.2) into (3.52) and (3.53), we obtain

$$U = \frac{\rho_1}{\rho_2 h}\left(\frac{c^2 - c_0^2}{c_0^2}\right), \quad V = \frac{3\rho_1}{2\rho_2 h^2}. \tag{5.4}$$

The general non-linear equation (3.51) for stationary waves therefore takes the form

$$\rho_1\left\{-\left(\frac{c^2 - c_0^2}{c_0^2}\right)f + \frac{3}{2}\frac{f^2}{h}\right\} = \rho_2 h\mathscr{F}\{f\}, \tag{5.5}$$

which agrees with (2.7) to the prescribed order of approximation. Hence, according to (3.78) and (3.79), the solitary-wave solution

$$f(x) = \frac{a\lambda^2}{x^2 + \lambda^2}$$

applies in this case with

$$c^2 = c_0^2\left(1 + \frac{3a}{4h}\right) \tag{5.6}$$

and

$$\lambda = \frac{4\rho_1}{3\rho_2}\frac{h^2}{a}. \tag{5.7}$$

Note that the vertical displacements in the lower fluid are given by (3.48) as $z = (\eta/h)f(x)$, while in the upper fluid they are given by (3.83).

These results also apply, with obvious modifications, when the layer of depth h is uppermost and is bounded on top by a rigid plane. If ρ_1 now denotes the density of the lighter fluid, the only change in the previous equations is that $-g$ replaces g in (5.3).

Next, let us suppose that the upper boundary is *free*. The solution of (5.1) then needs to satisfy (4.5), and so we have

$$\phi_0 = \frac{\eta-(c_0^2/g)}{h-(c_0^2/g)} \tag{5.8}$$

instead of (5.2). Here η is measured downwards, and correspondingly

$$\rho c_0^2(d\phi_0/d\eta)+g\rho\phi_0$$

is the quantity to be made continuous across the interface. Hence the argument used previously leads now to

$$c_0^2 = \frac{(\rho_2-\rho_1)gh}{\rho_2} = \frac{\rho_1}{\rho_2}\bar{c}_0^2, \tag{5.9}$$

where $\bar{c}_0^2$ is the value obtained in the case of a fixed upper boundary. This result agrees with one given by Lamb (1932, p. 372, equation (20)), in which taking the limit $kh \to 0$ establishes the comparison. Note that the effect of the free surface reduces the long-wave speed in the ratio $(\rho_1/\rho_2)^{\frac{1}{2}}$.

The formulae (3.52) and (3.53) now give

$$U = \frac{\rho_2}{\rho_1 h}\left(\frac{c^2-c_0^2}{c_0^2}\right), \quad V = \frac{\rho_1^2}{4\rho_1^2 h^2}. \tag{5.10}$$

Hence the solitary-wave properties are

$$c^2 = c_0^2\left(1+\frac{3\rho_2 a}{4\rho_1 h}\right) \tag{5.11}$$

and

$$\lambda = \frac{4}{3}\left(\frac{\rho_1}{\rho_2}\right)^2\frac{h^2}{a}. \tag{5.12}$$

Note from (5.12) as compared with (5.7) that the length of the wave, for a given amplitude, is decreased in the ratio $(\rho_1/\rho_2)^3$ by the effect of the free surface. Note also from (5.8) that there is a plane of zero displacement at a depth

$$c_0^2/g = h(\rho_2-\rho_1)/\rho_2$$

below the undisturbed level of the free surface: above this plane the fluid is elevated by a solitary wave, while below it the fluid is depressed. The wave is, of course, wholly of depression when the upper boundary is fixed.

Example 2. Bottom layer with exponential density variation

The model is illustrated in figure 8. We take $\rho(\eta) = \bar{\rho}\,e^{\beta(h-\eta)}$ in $(0,h)$, and $\rho(\eta) = \bar{\rho}\,(\text{const.})$ in (h,∞). From (3.13), the equation for ϕ_0 becomes

$$\frac{d^2\phi_0}{d\eta^2}-\beta\frac{d\phi_0}{d\eta}+\frac{\beta g}{c_0^2}\phi_0 = 0, \tag{5.13}$$

and hence the solution satisfying $\phi_0(0) = 0$, $\phi_0(h) = 1$ is

$$\phi_0 = \frac{e^{\frac{1}{2}\beta(\eta-h)}\sin q\eta}{\sin qh}, \tag{5.14}$$

with

$$q^2 = \frac{g\beta}{c_0^2} - \frac{\beta^2}{4}. \tag{5.15}$$

The remaining condition, $d\phi_0/d\eta = 0$ at $\eta = h$, gives

$$q\cot qh + \tfrac{1}{2}\beta = 0, \tag{5.16}$$

which determines q and thereupon c_0.

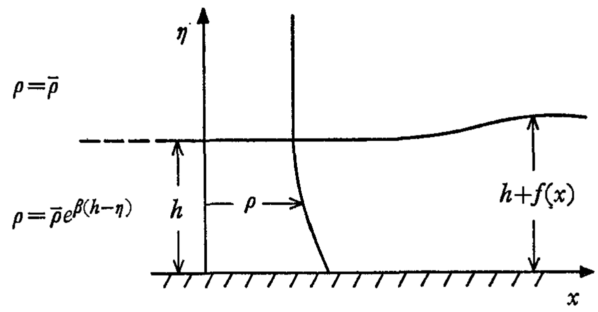

FIGURE 8. Illustration of system in which density decreases exponentially with height in bottom layer.

The notation $q^{(n)}$ $(n = 1, 2, 3, \ldots)$, as used in § 3.2, is now introduced to denote the solutions of (5.16) in order of increasing size. For small βh, we find from (5.16) that

$$q^{(n)}h = (n-\tfrac{1}{2})\pi + \frac{\beta h}{(2n-1)\pi} - O(\beta^2h^2), \tag{5.17}$$

and hence from (5.15) that

$$c_0^{(n)} = \frac{(g\beta)^{\frac{1}{2}}h}{(n-\frac{1}{2})\pi}\left\{1 - \frac{2\beta h}{(2n-1)^2\pi^2} + O(\beta^2h^2)\right\}. \tag{5.18}$$

In a practical application of this model, βh would probable be very small, so that just the first terms of (5.17) and (5.18) would be an adequate approximation. The result then is the same as what we would get by approximating the density distribution as a linear function, $\rho = \bar{\rho}\{1+\beta(h-\eta)\}$, and making the Boussinesq approximation.

The substitution of (5.14) into (3.52) gives

$$\frac{Uc_0^2}{c^2-c_0^2} = \operatorname{cosec}^2 qh\int_0^h (q\cos q\eta + \tfrac{1}{2}\beta\sin q\eta)^2\,d\eta$$
$$= \tfrac{1}{2}q^2h(1+\delta^2)^2 + \tfrac{1}{4}\beta(1+\delta^2), \tag{5.19}$$

where $\delta = \frac{1}{2}\beta/q$, $1+\delta^2 = \operatorname{cosec}^2 qh$, and where the mode number is implicit. Next, (3.53) leads to

$$V = \tfrac{3}{2}e^{-\frac{1}{2}\beta h}\operatorname{cosec}^3 qh\int_0^h e^{\frac{1}{2}\beta\eta}(q\cos q\eta + \tfrac{1}{2}\beta\sin q\eta)^3\,d\eta$$
$$= \frac{3q^2(1+\delta^2)}{9+\delta^2}\{(-1)^{n+1}(3+2\delta^2) - \delta(1+\delta^2)^{\frac{1}{2}}(1-e^{-\frac{1}{2}\beta h})\}. \tag{5.20}$$

For $\beta h \ll 1$ and therefore $\delta \ll 1$, the first approximation to these results is

$$\left[\frac{Uc_0^2}{c^2-c_0^2}\right]^{(n)} = \tfrac{1}{2}\{q^{(n)}\}^2 h = \frac{(n-\frac{1}{2})^2\pi^2}{2h}, \tag{5.21}$$

$$V^{(n)} = (-1)^{n+1}\{q^{(n)}\}^2 = \frac{(-1)^{n+1}(n-\frac{1}{2})^2\pi^2}{h^2}, \tag{5.22}$$

which evidently improves in accuracy with increasing n.

Note that $V^{(n)}$ and $\phi_0^{(n)}(h)$ are either both positive or both negative. This means that in a solitary wave the displacements at the top of the heterogeneous region are always upwards, whatever the mode number n, although for $n > 1$ there are strata in which the displacements are downwards. Hence, to a first approximation for small βh, the solitary-wave solution in $0 \leqslant \eta \leqslant h$ can be expressed as

$$z = (-1)^{n+1}\sin\left\{\frac{(2n-1)\pi\eta}{2h}\right\}\frac{a\lambda^2}{x^2+\lambda^2}, \tag{5.23}$$

with $a > 0$ and

$$c^{(n)} = c_0^{(n)}\left\{1+\frac{a}{h}\right\}^{\frac{1}{2}}, \tag{5.24}$$

$$\lambda = \frac{2}{(n-\frac{1}{2})^2\pi^2}\frac{h^2}{a}. \tag{5.25}$$

Corresponding expressions free from restriction on the magnitude of βh can be written down directly from (5.19) and (5.20).

Example 3. *Density variation with* tanh *profile*

The model is illustrated in figure 9. The density is given by

$$\rho(\eta) = \bar{\rho}(1-\varpi\tanh\alpha\eta), \tag{5.26}$$

and the fluid is supposed to extend infinitely far both upwards and downwards. This example is of special interest in that the bounds of the heterogeneous region I are not precisely defined, although the way to apply the results of the main analysis is clear. We might reason, in the first place, that there could be no significant effect on the possible wave motions if the system were very slightly modified so that the density variation were terminated at $\eta = \pm h$, say, such that $\alpha h \gg 1$. Then the same formal approach as used previously would be applicable. But clearly the final results would be practically independent of the exact choice of h, and this device would introduce needless complications. The more satisfactory rationale is to regard the two expressions (3.48) and (3.50) for $z(x,\eta)$ as inner and outer approximations, valid respectively for $|\eta| \ll \lambda$ and $\alpha^{-1} \ll |\eta|$. The long-wave assumption underlying the theory amounts here to the statement that $\alpha\lambda = O(\epsilon^{-1}) \gg 1$, and so we may envisage an intermediate region, $\alpha^{-1} \ll |\eta| \ll \lambda$, in which the two approximate solutions overlap. The method of analysis developed in §3 now makes the two solutions give the same values of z and $\partial z/\partial\eta$ throughout this region, rather than at a specific interface $\eta = h$. The details of the analysis remain just the same as before, except that the upper boundary condition on ϕ_0 may conveniently be taken to apply at $\alpha\eta = \infty$.

As was explained in § 4.3, the Boussinesq approximation will be used in the treatment of this example, and only antisymmetric wave modes will be considered. Accordingly, the other condition on ϕ_0 is that $\phi_0(0) = 0$ or, which is the same thing, that ϕ_0 is an odd function of η.

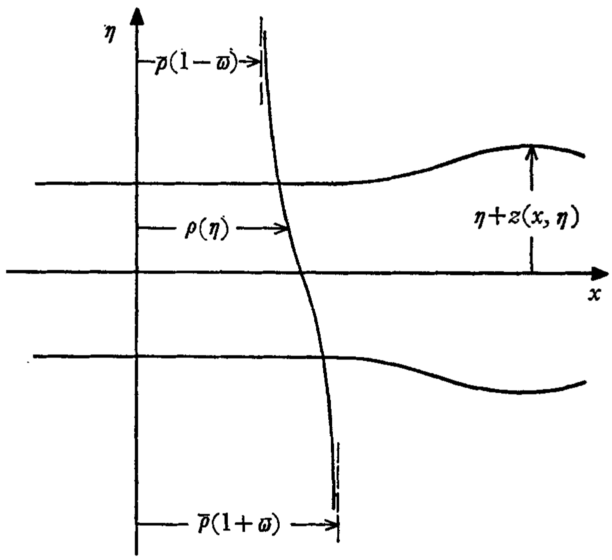

FIGURE 9. Illustration of system in which the heterogeneous layer is remote from any boundary. The asymptotic values of the density, $\bar{\rho}(1 \pm \varpi)$, are indicated.

From (4.14) with (5.26) substituted, the equation for ϕ_0 becomes

$$\frac{d^2\phi_0}{d\eta^2} + \left(\frac{g\varpi\alpha\,\mathrm{sech}^2\,\alpha\eta}{c_0^2}\right)\phi_0 = 0. \tag{5.27}$$

Now put

$$\left.\begin{aligned} &\phi_0 = P(\mu), \\ \text{with}\quad &\mu = \tanh\alpha\eta, \quad d\mu = \alpha(1-\mu^2)\,d\eta. \end{aligned}\right\} \tag{5.28}$$

Then (5.27) takes the form

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP}{d\mu}\right\} + \left(\frac{g\varpi}{\alpha c_0^2}\right)P = 0, \tag{5.29}$$

which is seen to be *Legendre's equation* if

$$g\varpi/\alpha c_0^2 = n(n+1), \tag{5.30}$$

where n is a positive integer. The condition (5.30) is necessary for (5.29) to have a single-valued solution applicable over the whole physical domain $-1 \leqslant \mu \leqslant 1$ (Whittaker & Watson 1927, § 15.2), and we conclude that it must fix the possible values of the wave speed c_0. Thus, rearranging (5.30), we have

$$c_0^{(n)} = \left\{\frac{g\varpi}{n(n+1)\alpha}\right\}^{\frac{1}{2}}; \tag{5.31}$$

and the corresponding solutions that remain finite throughout $-1 \leqslant \mu \leqslant 1$ are

$$\phi_0^{(n)} = P_n(\mu) = P_n(\tanh\alpha\eta), \tag{5.32}$$

where P_n denotes the Legendre polynomial of degree n. Only the solutions with *n odd* are relevant at the moment, however, because only these have the required symmetry property, so satisfying $\phi_0^{(n)}(0) = P_n(0) = 0$. The other end condition, $d\phi_0^{(n)}/d\eta \to 0$ for $\eta \to \infty$, is obviously satisfied. We note also that the Legendre polynomials as always defined (Whittaker & Watson, § 15.1) give us the further requirement $\phi_0^{(n)}(\infty) = 1$, since $P_n(1) = 1$.

In the formula (3.52) for U the upper limit of integration may now be taken as $\alpha\eta = \infty$ and, in accordance with the remarks made at the end of § 4.3, the approximation $\rho = \bar{\rho}$ may be made in the integrand. The substitution of (5.32) then gives

$$\left[\frac{Uc_0^2}{c^2-c_0^2}\right]^{(n)} = \alpha\int_0^1 (1-\mu^2)\left(\frac{dP_n}{d\mu}\right)^2 d\mu.$$

Integrating by parts and using the differential equation for P_n, we hence obtain

$$\left[\frac{Uc_0^2}{c^2-c_0^2}\right]^{(n)} = \alpha n(n+1)\int_0^1 P_n^2\, d\mu = \frac{\alpha n(n+1)}{2n+1}, \tag{5.33}$$

where the final equation is a fundamental result in the theory of the Legendre polynomials (Whittaker & Watson, § 15.14). Similarly, the formula (3.53) gives, for $n = 1, 3, 5, \ldots$,

$$V^{(n)} = \tfrac{3}{2}\alpha^2\int_0^1 (1-\mu^2)^2\left(\frac{dP_n}{d\mu}\right)^3 d\mu = \frac{6\alpha^2 n^2(n+1)}{3n+2}\int_0^1 P_n^2 P_{n-1}\, d\mu$$
$$= 6\alpha^2\left\{\frac{1.3\ldots n}{2.4\ldots(n-1)}\right\}^3\left\{\frac{2.4\ldots(3n-1)}{1.3\ldots(3n+2)}\right\}. \tag{5.34}$$

Here the second expression is easily obtainable from the first by integrations by parts and use of recurrence formulae for P_n, and to obtain the last a formula given by Hobson (1931, p. 87) may be used. The first two values specified by (5.34) are

$$V^{(1)} = \tfrac{4}{5}\alpha^2, \quad V^{(3)} = \tfrac{288}{385}\alpha^2. \tag{5.35}$$

In the first mode, since $P_1(\mu) = \mu$, the solitary-wave solution for the central region ($|\eta| \ll \lambda$) is therefore

$$z = (\tanh\alpha\eta)\frac{a\lambda^2}{x^2+\lambda^2}, \tag{5.36}$$

with

$$[c^{(1)}]^2 = \frac{g\varpi}{2\alpha}\{1+\tfrac{3}{5}\alpha a\}, \tag{5.37}$$

$$\lambda = 5/(2\alpha^2 a). \tag{5.38}$$

This solution applies, of course, both above and below the plane of symmetry. Above, the displacements are upwards, while below they are downwards.

For the higher modes also, the $V^{(n)}$ are all positive, which means that the displacements in the outer parts of the central region (i.e. where $\mu \simeq \pm 1$), and hence also in regions II, are always away from the plane of symmetry. In the next higher mode, for instance, the solitary-wave solution for the central region can be expressed as

$$z = \tfrac{1}{2}(5\tanh^3\alpha\eta - 3\tanh\alpha\eta)\frac{a\lambda^2}{x^2+\lambda^2}, \tag{5.39}$$

with $a > 0$ and

$$[c^{(3)}]^2 = \frac{g\varpi}{12\alpha}\{1+\tfrac{12}{55}\alpha a\}, \tag{5.40}$$

$$\lambda = \frac{385}{144\alpha^2 a}. \tag{5.41}$$

Note from (5.39) that the displacements are inwards (i.e. downwards above and upwards below $\eta = 0$) for $\alpha|\eta| < |\tanh^{-1}(3/5)^{\frac{1}{2}}| = 1{\cdot}022$.

As a final comment on this example, let us specify the coefficients of the approximate dispersion relation (3.17) for *infinitesimal* sinusoidal waves. The integral appearing in (3.16) has already been evaluated in the derivation of (5.33). Hence, using the expression (5.31) for $c_0^{(n)}$, we obtain at once

$$c^{(n)} = \left\{\frac{g\varpi}{n(n+1)\alpha}\right\}^{\frac{1}{2}}\left\{1-\frac{2n+1}{2n(n+1)}\frac{|k|}{\alpha}+O(k^2/\alpha^2)\right\}. \tag{5.42}$$

This result is found to hold also for the even modes, with the obvious exception of $n = 0$, and it agrees with the exact result for $c^{(n)}(k)$ found by Groen (1948) for this example. [In the exceptional case the heterogeneous layer moves up or down as a whole, and a study of the full linearized equations shows that $c^2 = O(\varpi g/k)$ rather than $O(\varpi g/\alpha)$, so that a long-wave approximation of the present kind is clearly unavailable. Moreover, the Boussinesq approximation appears to be invalid for this particular wave mode (cf. Groen 1948).] The reasons why no attempt is made to develop a finite-amplitude theory for the even modes have been explained at the end of § 4.3.

6. Conclusion

It is worth emphasis that only a first approximation to the effects of finite wave amplitude has been worked out, and the present results may be unreliable quantitatively when the amplitude is not a fairly small fraction of h (or α^{-1} in the last example). The salient properties of this class of internal waves seem to be well explained, however; and though higher approximations could be derived readily enough by following the scheme set out in § 3.3, there would be comparatively little profit in such extensions.To deal with waves of large amplitude, direct numerical solution of the unsimplified governing equations (e.g. equation (3.3), or some transformation of it) is likely to be more rewarding. With increasing amplitude, the ultimate limitation on perturbation analyses of the present kind is posed by the occurrence of stagnation points within the flow, which develop at still larger amplitudes into regions of circulating fluid (see Benjamin 1967 for a discussion of this point). In this situation the transformation (3.1), upon which our whole analysis is based, is certainly no longer valid—since it presupposes a non-singular mapping of the *whole* flow field by the original streamlines of the undisturbed flow; and indeed the situation offers small hope of any general analytical treatment.

The main results obtained in §§ 3 and 4 need only minor modifications to include the effects of a horizontal current in region I. A redevelopment of the analysis is hardly required, since the outcome has several precedents in previous papers by the author dealing with wave propagation in non-uniform flows

(Benjamin 1962, 1966, 1967). Let $U(\eta)$ denote the velocity of the undisturbed fluid in the x-direction, and suppose that $U = 0$ outside region I. Then, if c is the wave speed, the function $W = U \pm c$ represents the primary velocity as observed in a frame of reference travelling with the wave, and $W_0 = U \pm c_0$ may be written to denote the corresponding function for infinitesimal waves of extreme length. Here the ambiguous sign covers the case of propagation with $(-)$ and against $(+)$ the current. Reference being made to the previous papers for explanation, the modifications may now be listed as follows: (i) in the equation (3.13) for ϕ_0, after multiplication of the two terms by c_0^2 the factor ρc_0^2 in the first should be replaced by ρW_0^2. (ii) In the free-surface condition (4.5), c_0^2 should be replaced by $W_0^2(0)$. (iii) In the formula (3.52) for U, the factor $(c^2 - c_0^2)\rho$ should be replaced in the integrand by $(W^2 - W_0^2)\rho$. (iv) In the formula (3.52) for V, the factor ρ should be replaced by $\rho W_0^2/c_0^2$.

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Appendix

Here an alternative expression for $\mathscr{F}\{f(x)\}$ is noted, applying in the case $f(\pm\infty) = 0$. Admittedly there is little use in this as regards the present problem, since the exact solutions of equation (3.51) have already been found. But $\mathscr{F}$ will also be encountered in problems of unsteady wave motion, where equation (1.11) would apply, and it is possible that the alternative expression might then be more helpful than the previous one in terms of Fourier integrals.

First we observe from (3.47) and (3.55) that the Fourier transform of $\mathscr{F}\{f(x)\}$ is

$$\begin{aligned} |k|\bar{f}(k) &= \frac{|k|}{2\pi}\int_{-\infty}^{\infty} f(w)\, e^{ikw}\, dw \\ &= \frac{i(\operatorname{sgn} k)}{2\pi}\int_{-\infty}^{\infty} \frac{df}{dw}\, e^{ikw}\, dw. \end{aligned} \tag{A 1}$$

But, by a simple application of contour integration, we have

$$\mathscr{P}\int_{-\infty}^{\infty} \frac{e^{ikw}}{z-w}\, dz = i\pi(\operatorname{sgn} k)\, e^{ikw}, \tag{A 2}$$

where $\mathscr{P}$ denotes the Cauchy principal value of the integral taken along the real axis. Hence (A 1) is expressible by

$$\frac{1}{2\pi^2}\int_{-\infty}^{\infty} \left\{\mathscr{P}\int_{-\infty}^{\infty} \frac{e^{ikw}}{z-w}\, dz\right\} \frac{df}{dw}\, dw. \tag{A 3}$$

The order of the integrations in (A 3) can be reversed, whereupon this expression is cast in the standard form of a Fourier transform (i.e. z exchanges roles with w as the dummy variable corresponding to x in (3.47)). Then it is seen that

$$\mathscr{F}\{f(x)\} = \frac{1}{\pi}\mathscr{P}\int_{-\infty}^{\infty} \frac{1}{x-w}\left(\frac{df}{dw}\right) dw, \tag{A 4}$$

which expresses $\mathscr{F}$ as the Hilbert transform of $-df/dx$. This result is also derivable, on a perhaps more familiar basis, from the solution of the original potential problem in terms of a Green's function (i.e. the problem defined by (3.21) and (3.22), also described below (1.10)).

It may be of interest to check that the solution (3.78) does in fact satisfy the alternative form of non-linear integral equation given by putting (A 4) in (3.51). We have in this case

$$\mathscr{F}\{f(x)\} = \frac{2a\lambda^2}{\pi}\mathscr{P}\int_{-\infty}^{\infty} \frac{w\,dw}{(w-x)\,(w^2+\lambda^2)^2},$$

from which there follows by fairly obvious steps

$$\mathscr{F}\{f(x)\} = \frac{2a\lambda^2}{\pi(x^2+\lambda^2)}\left[-\lambda^2\int_{-\infty}^{\infty}\frac{dw}{(w^2+\lambda^2)^2}+x\mathscr{P}\int_{-\infty}^{\infty}\frac{dw}{(w-x)\,(w^2+\lambda^2)}\right]. \quad \text{(A 5)}$$

The first integral in (A 5) is a standard form. The principal value of the second can be found at once by use of Cauchy's theorem on contour integration: by Jordan's lemma, this principal value is the sum of the residue for the pole at $w = i\lambda$ and half the residue for the pole at $w = x$. Hence a combination of the terms leads to

$$\mathscr{F}\{f(x)\} = \frac{a\lambda(\lambda^2-x^2)}{(x^2+\lambda^2)^2}, \quad \text{(A 6)}$$

which is, as required, the same as $-Uf+Vf^2$ when $U = 1/\lambda$ and $V = 2/\lambda a$.

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